ON THE OPTIMAL CONTROL OF A SYSTEM GOVERNED BY A LINEAR PARABOLIC EQUATION WITH 'WHITE NOISE' INPUTS

by

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1. Introduction

Let $\mathcal{L}$ be a smooth elliptic operator whose coefficients are defined on $\bar{D} \times [0,T]$, where $\bar{D} \in E^n$ (Euclidean n-space).

We consider the problem of the optimal control of systems with the formal representation

$$W_t = \mathcal{L}W + bu + \sum_i \sigma_i \xi_i \qquad (*)$$

satisfying $W(x,t) \to 0$ as $x \to \partial D$, $t \geq 0$, and with control

$$u(x,t) = \int k(v,x,t)W(v,t)dv$$

and cost criterion

$$C^u(\varphi,t) \equiv E^u_\varphi \int_t^T \int W(x,s)W(y,s)S(x,y,s)dxdyds$$

$$+ E^u_\varphi \int_t^T \int P(x,s)u^2(x,s)dxds,$$

where ξ_i is the formal derivative of the Wiener process $z_i(t)$ and E^u_φ is the expectation given the control u, and initial condition $\varphi(\cdot)$. A precise meaning is given to all terms in the sequel. An equation of the form $(*)$ seems like a useful model of a variety of noise disturbed objects. but it also arises in the following way. Suppose that an object is governed by $H_t = \mathcal{L}H + bu$ and the noise corrupted observations having the Itô differential (for each $x \in \bar{D}$) $dy_i(x,t) = dt\int m_i(v,x,t)H(v,t)dv + dw_i$ are taken, where the w_i are Wiener processes. Then, the conditional expection† $W(x,t) \equiv E\{H(x,t)|y(v,s),\ s \leq t,\ v \in \bar{D}\}$ has a representation in the form (*). In fact, (26) is the relevant 'Riccati' equation (with a reversed time parameter). The results herein concern the first boundary value problem for a single parabolic equation. However, it is clear that the method is applicable (and results easily extendible to) the second boundary value problem, or to a family of parabolic equations. The latter model is quite versatile. For example, we can use a vector parameter process $W(x,t)$ generated by (*) as the input to another system (e.g., $(\partial/\partial t-\mathcal{L})Y = W + b'u'$). We can thus generate the 'distributed system' analogs of the 'linear Gauss-Markov' processes, and treat the corresponding average quadratic cost control problem.

The results are based on the results of [1] which provide criterion which guarantee that there is a version of a vector parameter process which with probability are (w.p.l.) is continuous or differentiable in some particular parameters. Without these latter results, eq. (1) would lose its intuitive meaning (as would stochastic differential equations if the paths were not known to be continuous w.p.l.). Previous works concerned with 'random' partial differential equations [2], [3], were concerned with the nature of the random solution corresponding to a random, but

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†The filtering problem will be treated in a subsequent work. This paper is devoted strictly to the control problem.

smooth, boundary condition. Once the necessary smoothness properties of the process $W(x,t)$ are established, much of the analysis is similar to the analysis of the corresponding deterministic problem. However, to our knowledge the few treatments of the deterministic problem (e.g.; see the interesting reference [4]) are essentially formal in nature. Most of the proofs are slightly abbreviated. We have chosen to omit the details of several arguments dealing mainly with the smoothness properties of potentials and related integrals. The arguments are tedious and standard. Some are based on existence theorems (Lemma 2.3) and most others use the arguments of [5], Chapter 1, Section 3-5.

In Section 2, some needed results on processes with a vector parameter set are given. The proofs of the statements of Lemma 2.1 are found in [1]. The proofs of the statements of Lemma 2.2 and its corollary follow from Lemma 2.1 and the properties of stochastic integrals depending on a scalar time parameter. Theorems 3.1 and 3.2 define the solution of (*) and its basic properties; continuity w.p.1, existence of Hölder continuous w.p.1. second derivatives (with respect to the x_i), etc. The optimality and 'approximation in policy space' results appear in Section 4. Although we deal with a single 'white noise' input, the results are obviously valid for the no more general, 'infinite dimensional' white noise input of the Corollary to Lemma 2.2.

2. Mathematical Preliminaries

Definition. Following the usual usage, a version of the vector parameter scalar valued process $f(y)$ is any scalar valued process $\tilde{f}(y)$ such that $P\{\tilde{f}(y) = f(y)\} = 1$ for all vector parameter values y. The parameter y varies over a domain R in E^n.

Lemma 2.1. *Let $\bar{R}$ be the closure of a bounded open domain with generic point $y = (y_1, \ldots, y_n)$, and let the boundary of $\bar{R}$ have the property that any line intersects it only finitely often. Let $f(\cdot)$ be a stochastically continuous process with parameter set $\bar{R}$. Suppose that for each nonempty section $\bar{R}_i(b_1, \ldots, b_{i-1}, b_{i+1}, \ldots, b_n) \equiv \bar{R} \cap \{y: y_j = b_j,\ j \neq i\}$, there is a null ω set $N_i(b_1, \ldots, b_{i-1}, b_{i+1}, \ldots, b_n)$ so that some version of*† *$f(b_1, \ldots, b_{i-1}, \cdot, b_{i+1}, \ldots, b_n)$ is continuous on $\bar{R}_i(b_1, \ldots, b_{i-1}, b_{i+1}, \ldots, b_n)$ for $\omega \notin N_i(b_1, \ldots, b_{i-1}, b_{i+1}, \ldots, b_n)$. Then ([1] Theorem 1) there is a version of $f(y)$ which is continuous on $\bar{R}$ w.p.1.*

Let the mean square derivatives of $f(y)$, with respect to $y_1, \ldots, y_r$, have continuous versions on $\bar{R}$ w.p.1. and let there be a continuous version of $f(y)$ on $\bar{R}$ w.p.1. Then there is a version of $f(y)$ which, w.p.1., is continuous and has continuous ordinary derivatives with respect to $y_1, \ldots, y_r$, on $\bar{R}$ (denoted

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† $f(b_1, \ldots, b_{i-1}, \cdot, b_{i+1}, \ldots, b_n)$ is $f(\cdot)$ restricted to $\bar{R}_i(b_1, \ldots, b_{i-1}, b_{i+1}, \ldots, b_n)$.

by $(D_i f(\cdot))$, $i = 1,\dots,r)$ ([|], Theorem 4).

Under the condition ($|\delta|$ is the Euclidean norm)

$$E|f(y+\delta)-f(y)|^{\alpha} \leq |\delta|^{1+\beta} \qquad (**)$$

for some $\alpha > 0$, $\beta > 0$, there is a Hölder† continuous (with Hölder exponent α/β) version of $f(\cdot)$ on $\bar{R}$ w.p.1. ([1], Theorem 2). Note that the above implies that if $f(y)$ and the mean square derivatives with respect to $y_1,\dots,y_r$ satisfy (**), then a version of $f(\cdot)$ has continuous derivatives with respect to $y_1,\dots,y_r$, w.p.1.

An immediate consequence of Lemma 2.1 is

Lemma 2.2. Let $x \in \bar{D}$ and $s,t \in [0,T]$. Suppose $\bar{R} \equiv D \times [0,T]$ satisfies the conditions on $\bar{R}$ in Lemma 2.1. Let z_s be a Wiener process and suppose the function $\alpha(x,t,s)$ satisfies

$$\int_0^T \alpha^2(x,t,s)ds < \infty$$

for each $x \in \bar{D}$ and $t \in [0,T]$ and

†By Hölder continuity (with exponent $\gamma > 0$) we mean that $|f(y+\delta)-f(y)| \leq K(\omega)|\delta|^{\gamma}$ where $K(\omega) < \infty$ w.p.1. and does not depend on y. In works on partial differential equations, where we let $y = (x,t)$, Hölder continuity is meant to imply $|f(x,t)-f(y,s)| \leq K(\omega)(|x-y|^{\gamma}+|t-s|^{\gamma/2})$. Since we are not concerned with the specific value of γ, either form is suitable in this paper.

$$\int_0^t [\alpha(x,t+\Delta,s)-\alpha(x,t,s)]^2 ds \leq \epsilon(\Delta)$$
$$\int_0^t [\alpha(x+\delta,t,s)-\alpha(x,t,s)]^2 ds \leq \epsilon(|\delta|) \qquad (***)$$

where $\epsilon(\Delta) \to 0$ as $\Delta \to 0$. Then† the random function

$$\psi(x,t) = \int_0^t \alpha(x,t,s)dz_s$$

has a Hölder continuous version on $\overline{R}$ w.p.1., for some Hölder exponent $\gamma > 0$. If, further, the $D_i\alpha(x,t,s) \equiv \partial\alpha(x,t,s)/\partial x_i$ satisfy the conditions on $\alpha(x,t,s)$ above, then the random functions

$$(D_i\psi(x,t)) \equiv \int_0^t D_i\alpha(x,t,s)dz_s$$

can be identified with the mean square derivative of $\psi(x,t)$ with respect to x_i, and there is a version of $\psi(x,t)$ which is continuous and has Hölder continuous (some non-random exponent $\gamma > 0$)

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†The details can be deduced from Lemma 2.1. The reader is referred to [1], Example 2, for the details of a similar result. (***) implies (**) of Lemma 2.1 since $E\{\int_0^{t+\Delta}\alpha(x+\delta,t+\Delta,s)dz_s - \int_0^t \alpha(x,t,s)dz_s\}^{2m}$ $= (2m-1)\cdot\cdot 5.3.1 \; [\int_0^t (\alpha(x+\delta,t+\Delta,s)-\alpha(x,t,s))^2 ds]^m$ which is $O(|\Delta|^{1+\beta}+|\delta|^{1+\beta})$ for some $\beta > 0$ and a sufficiently large m.

derivatives (with respect to x_i*) on* $\bar{R}$ w.p.l. *If the second derivatives (with respect to the* x_i*) of* $\alpha(x,t,s)$ *satisfy the conditions on* $\alpha(x,t,s)$, *then the conclusions hold for the second derivatives of* $\psi(x,t)$ *with respect to the* x_i.

Corollary. Let z_{is}, $i = 1,\ldots,$ *be a sequence of independent Wiener processes on* $[0,T]$ *and let the family* $\alpha_i(x,t,s)$ *satisfy*

$$\int_0^T \sum_i \alpha_i^2(x,t,s)ds < \infty, \quad \int_0^T \sum_i [\alpha_i(x+\delta,t,s)-\alpha(x,t,s)]^2 ds \leq \epsilon(|\delta|)$$

$$\int_0^t \sum_i [\alpha_i(x,t+\Delta,s)-\alpha_i(x,t,s)]^2 ds \leq \epsilon(\Delta),$$

where $\epsilon(\Delta) \to 0$ *as* $\Delta \to 0$. *Then* w.p.l. *there is a Holder continuous version of*

$$\psi(x,t) = \sum_i \int_0^t \alpha_i(x,t,s)dz_{is}.$$

The remainder of the statements of Lemma 2.2 have their obvious analogs here.

Define D_i as the differential operator $\partial/\partial x_i$ and let $U_t = \partial U/\partial t$ and

$$\mathcal{L} = \sum_{i,j} a_{ij}(x,t)D_iD_j + \sum_i d_i(x,t)D_i + c(x,t).$$

Let us collect the following assumptions here. Note that Holder continuity on the compact set $\overline{R}$ is equivalent to uniform Hölder continuity on $\overline{R}$.

(E-1) Let $R = D \times [0,T]$, where D is a bounded and open Borel measurable domain. To each point on ∂D, let there exist a neighborhood V and a function on $h(\cdot)$ so that $\partial D \cap V$ has the representation $x_i = h(x_i,\ldots,x_{i-1},x_{i+1},\ldots,x_n)$ for some component x_i, where $h(x)$ has Hölder continuous second partial derivatives[†].

(E-2) On $\overline{R}$, the coefficients of $\mathcal{L}$ are bounded and Hölder continuous.

(E-3) There exists a real number $K > 0$ so that

$$\sum_{i,j} a_{ij}(x,t)\xi_i\xi_j \geq K \sum_i \xi_i^2$$

for any vector ξ.

(E-4) Let $\sigma(x,t), D_i\sigma(x,t), D_iD_j\sigma(x,t)$ be Hölder continuous in $\overline{R}$.

(E-5) Let $\sigma(x,t)$ and $\mathcal{L}\sigma(x,t)$ tend to zero as $x \to \partial D$ in $\overline{R}$

(E-6) Let $b(x,t)$ be Hölder continuous on $\overline{R}$ and $k(y,x,t)$ be bounded, measurable and Hölder continuous in x,t on $\overline{R}$, uniformly in y.

(E-7) Let $a_{ij}(x,t)$ have Hölder continuous second derivatives in $\overline{R}$, and $d_i(x,t)$ Hölder continuous first derivatives in $\overline{R}$.

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† (E-1) implies the condition on $\overline{R}$ of Lemma 2.1.

It will be helpful to collect the following results here. They will be used without reference in the sequel.

Lemma 2.3A ([5], Theorem 7, Chapter 3). Suppose (E-1)-(E-3). Let $\varphi(x,0)$ have Hölder continuous second derivatives in $\bar{D}$ and satisfy $\varphi(x,0) \to 0$ and $\mathcal{L}\varphi(x,0) \to 0$ as $x \to \partial D$. Let $\varphi(\partial D, t) \equiv 0$. Suppose that $f(x,t)$ is Hölder continuous in $\bar{R}$ and tends to zero as $x \to \partial D$. Then there is a continuous unique solution to $U_t = \mathcal{L}U + f$ on $\bar{R}$ which satisfies the boundary condition $\varphi(x,t)$. $U_t(x,t), D_iU(x,t)$ and $D_iD_jU(x,t)$ are Hölder continuous on $\bar{R}$ and $\mathcal{L}U(x,t) \to 0$ as $x \to \partial D$, $t \geq 0$.

Define the Banach space $\bar{C}^0_\alpha$ of functions on $\bar{R}$ which satisfy $f(x,t) \to 0$ as $x \to \partial D$ and with the norm

$$\|f\|_\alpha = \sup_{x,t\in\bar{R}} |f(x,t)| + \sup_{\substack{x,t\in\bar{R}\\ y,s\in\bar{R}}} \frac{|f(x,t)-f(y,s)|}{|x-y|^\alpha+|t-s|^{\alpha/2}} .$$

Let $\bar{C}^0_{2+\alpha}$ be the sub (Banach) space of $\bar{C}^0_\alpha$ of functions which satisfy $\mathcal{L}f(x,t) \to 0$ as $x \to \partial D$, and have the norm

$$\|f\|_{2+\alpha} = \|f\|_\alpha + \sum_i \|D_i f\|_\alpha + \sum_{i,j} \|D_iD_jf\|_\alpha.$$

Then, with homogeneous boundary conditions, the equation $U_t - \mathcal{L}U = f$ represents a continuous linear map of $\bar{C}^0_\alpha$ into $\bar{C}^0_{2+\alpha}$. (Follows from [5], Chapter 3, eqn. 2.21, and the first part of the Lemma.)

Lemma 2.3B ([5], Theorem 16, Chapter 3). Suppose (E-1)-(E-3). Then there is a Greens function $G(x,x';t,t')$ for $U_t = \mathcal{L}U$. $G(x,x';t,t') \to 0$, as $x \to \partial D$ for $t > t'$. $D_i G(x,x';t,t')$, $D_i D_j G(x,x';t,t')$ and $G_t(x,x';t,t')$ are continuous in (x,t) on $D \times (t',T]$. If $\sigma(x,t)$ is continuous in x for each t, then $\alpha(x,t,t') \equiv \int G(x,x';t,t')\sigma(x',t')dx'$ satisfies $(\partial/\partial t - \mathcal{L})\alpha(x,t,t') = \sigma(x,t)$ on $D \cap (t',T]$ and tends to zero as $x \to \partial D$ for $t > t'$. If $\sigma(x,t)$, and $\mathcal{L}\sigma(x,t)$ tend to zero as $x \to \partial D$ and are Holder continuous on $\bar{R}$, then $\alpha(x,t,t')$ satisfies the same conditions for $(x,t) \in \bar{D} \cap [t',T]$. If $\sigma(x,t)$ is bounded and measurable, then $\int_0^t dt'\alpha(x,t,t')$ tends to zero as $x \to \partial D$ and is continuous on $\bar{R}$. (The last statement follows from the arguments concerning potentials in [5], Chapter 1, Sections 3-5.)

Lemma 2.3C ([5], Theorem 17, Chapter 3). Suppose (E-1)-(E-3) and (E-7). Then the Greens function for the adjoint operator $\partial/\partial t + \mathcal{L}^*$ is

$$G^*(x,x';t,t') = G(x',x;t',t).$$

Note that $G^*(x,x';t,t')$ is defined for $t < t'$ and that $(\partial/\partial t + \mathcal{L}^*)G^*(x,x';t,t') = 0$ on $D \times [0,t')$.

3. The Stochastic Partial Differential Equation

Theorem 3.1. *Suppose* (E-1)-(E-5). *Let* $\varphi(x)$ *have Holder continuous second derivatives on* $\bar{R}$ *and tend to zero as* $x \to \partial D$. *Let* $\mathcal{L}\varphi(x) \to 0$ *as* $x \to \partial D$. *Then there is a process* $W(x,t)$ *with parameter set* $\bar{R}$ *satisfying* (z' *denotes* $z(t')$)

$$W(x,t) = \int G(x,x';t,0)\varphi(x')dx' + \int_0^t dz'\{\int G(x,x';t,t')\sigma(x',t')dx'\}. \tag{1}$$

There is a version of $W(x,t)$ *which is continuous on* $\bar{R}$ w.p.1. *and satisfies* $W(x,0) = \varphi(x)$, $W(\partial D,t) = 0$. $D_i W(x,t)$ *and* $D_i D_j W(x,t)$ *are also Holder continuous on* $\bar{R}$ w.p.1. *and* $\mathcal{L}W(x,t) \to 0$ *as* $x \to \partial D$. *Also,* w.p.1.

$$W(x,t) = \int G(x,x';t,s)W(x',s)dx' + \int_s^t dz' \int G(x,x';t,t')\sigma(x',t')dx'. \tag{2}$$

For each fixed x *in* $\bar{D}$, $W(x,t)$ *has the Itô differential*

$$dW = \mathcal{L}W(x,t)dt + \sigma(x,t)dz. \tag{3}$$

Remark. $W(x,t)$ is not (w.p.1.) differentiable in t. The smoothness of $\sigma(x,t)$ determines the smoothness of $W(x,t)$. Lemma 2.2 plays a crucial role here. It is not a priori obvious that the last term in (1) has a version which is sufficiently smooth w.p.1. Lemmas 2.1, 2.2 turn the estimates of the stochastic continuity of (1) and the stochastic continuity of its first two mean square derivatives (with respect to the x_i) into a statement concerning the existence of a version for each x, t, so that the actual sample functions are smooth. The order of integration in the stochastic integral in (1) must be preserved. Also, the theorem implies that the $\mathcal{L}W$ term in (3) has a version which is continuous w.p.1. on $\overline{R}$.

It can be shown that the process $W(\cdot,t)$. with parameter t, is a continuous Markov process with values in a Banach space of functions which satisfy the appropriate boundary conditions $(W(x,t) \to 0$ and $\mathcal{L}W(x,t) \to 0$ as $x \to \partial D)$ and have Hölder continuous second derivatives (for some fixed non-random Hölder exponent).

Proof. The proof is a consequence of Lemmas 2.2 and 2.3. Let $\beta(x,t)$ be the first term on the right of (1). Then (Lemma 2.3) $\mathcal{L}\beta = \beta_t$; $\beta(x,t)$ satisfies the boundary conditions and $\mathcal{L}\beta(x,t) \to 0$ as $x \to \partial D$ for $t \geq 0$. $\beta(x,t)$ and $\mathcal{L}\beta(x,t)$ tend to $\varphi(x)$ and $\mathcal{L}\varphi(x)$, resp. as $t \to 0$. Write

$$\alpha(x,t,t') = \int G(x,x';t,t')\sigma(x',t')dx'.$$

Then for $t \geq t'$ (t' fixed) $\alpha(x,t,t')$, $D_i\alpha(x,t,t')$ and $D_iD_j\alpha(x,t,t')$, are Hölder continuous on $\overline{R}$ and each satisfies the conditions on $\alpha(x,t,t')$ of Lemma 2.2. Hence, by Lemma 2.2, there is a version of

$$\psi(x,t) = \int_0^t dz'\alpha(x,t,t')$$

which is Holder continuous w.p.l. on $\bar{R}$, and which has Holder continuous second derivatives with respect to the x_i w.p.l. on $\bar{R}$. $D_i\psi(x,t)$ and $D_iD_j\psi(x,t)$ can be identified with the Holder continuous versions (which exist w.p.l. in $\bar{R}$) of

$$D_i\psi(x,t) = \int_0^t dz' D_i\alpha(x,t,t') = \int_0^t dz' \int D_i G(x,x';t,t')\sigma(x',t')dx'$$

$$D_iD_j\psi(x,t) = \int_0^t dz' D_iD_j\alpha(x,t,t') = \int_0^t dz' \int D_iD_j G(x,x';t,t')\sigma(x',t')dx',$$

resp. Thus $\mathcal{L}\psi(x,t)$ has a version which is Holder continuous w.p.l., and which is clearly a Holder continuous version of $\int_0^t dz'\mathcal{L}\alpha(x,t,t')$. Using the continuity of $\mathcal{L}\psi(x,t)$ and the fact that $\mathcal{L}\psi(x,t) \to 0$ in probability (since $\mathcal{L}\alpha(x,t,t') \to 0$) as $x \to \partial D$, we have $\mathcal{L}\psi(x,t) \to 0$ as $x \to \partial D$. Thus (1) satisfies the required boundary conditions. Equation (3) follows from the definition of the Itô differential of (1) for each fixed x, and the observation that $d\int_0^t dz'\alpha(x,t,t') = dz\alpha(x,t,t) + dt\int_0^t dz'\alpha_t(x,t,t')$ where $\alpha(x,t,t) = \sigma(x,t)$ and $\alpha_t(x,t,t') = \mathcal{L}\alpha(x,t,t')$.

Equation (2) is obviously true w.p.l. for each fixed x,t,s. To show that it is true w.p.l. on $\bar{D} \times [0,T] \times [0,T]$, note first that

$$\psi(x,t,s) \equiv \int_s^t dz' \{\int G(x,x';t,s')\sigma(x',s')dx'\}$$

can be defined (using previous arguments) to be continuous (as a function of x,t,s) on $\bar{D} \times [0,T] \times [0,T]$, except for ω in some null set N. Define $\psi(x,t)$ as the (unique) $\lim_{s \to 0} \psi(x,t,s)$. The limit exists for $\omega \notin N$, and is a version of the continuous (for $\omega \notin N$) function $\psi(x,t)$ defined previously. Then, for $\omega \notin N$,

$$\psi(x,t) = \psi(x,s) + \psi(x,t,s).$$

Now writing (1) in the equivalent form

$$\begin{aligned} W(x,t) &= \int G(x,x';t,0)\varphi(x')dx' \\ &\quad + \int G(x,x';t,s)\psi(x',s,0)dx' + \psi(x,t,s) \end{aligned}$$

and using the semigroup property

$$\int G(x,x';t,0)\rho(x')dx' = \int G(x,x'';t,s)dx'' \int G(x'',x';s,0)\rho(x')dx'$$

and the continuity w.p.l. of $\psi(x,t,s)$ gives (2). Q.E.D.

The concern of the paper is restricted to systems with controls, which are linear in $W(x,t)$ and which appear linearly in the differential equation. Thus adding a control term to (3) we have, formally,

$$dW = \mathcal{L}W(x,t)\cdot dt + \sigma(x,t)dz + b(x,t)u(x,t) \tag{3a}$$

where the control is

$$u(x,t) = \int W(y,t)k(y,x,t)dy \tag{4}$$

Theorem 3.2 gives meaning to (3a).

Theorem 3.2. Suppose (E-6) and all the assumptions of Theorem 3.1, and let $u(x,t)$ be given by (4). Then there is a Hölder continuous (w.p.1) version of (5)

$$\begin{aligned} W(x,t) = \int G(x,x';t,0)\varphi(x')dx' &+ \int_0^t dz'\int G(x,x';t,t')\sigma(x',t')dx' \\ &+ \int_0^t dt'\int G(x,x';t,t')b(x',t')u(x',t')dx' \end{aligned} \tag{5}$$

Furthermore (w.p.1.) $D_iW(x,t)$ and $D_iD_jW(x,t)$ are Hölder continuous on $\bar{R}$, and both $W(x,t)$ and $\mathcal{L}W(x,t)$ tend to zero as $x \to \partial D$.

There is a kernel $B(x,x';t,t')$, so that

$$W(x,t) = \int_0^t dt'\int B(x,x';t,t')q(x',t')dx' \tag{6}$$

where $q(x,t)$ is the sum of the first two terms on the right side

<u>of</u> (5). $B(x,x';t,t')$ <u>maps</u>† $\overline{C}^{o}_{2+\alpha}(\overline{R})$ into $\overline{C}^{o}_{2+\alpha}(\overline{R})$. $W(x,t)$ has <u>the Itô differential</u>

$$dW(x,t) = \mathcal{L}W(x,t)dt + \sigma(x,t)dz + b(x,t)\int k(y,x,t)W(y,t)dydt \quad (7)$$

<u>Proof</u>. Let $W(x,t)$ be Hölder continuous on $\overline{R}$ (exponent α) and tend to zero as $x \to \partial D$. Let the Hölder exponent in (E-6) be $\gamma \geq \alpha$. Then

$$f(x',t') = b(x',t')\int W(y,t')k(y,x',t')dy \quad (8)$$

is Hölder continuous on $\overline{R}$ (exponent α) and tends to zero as $x' \to \partial D$. Thus, (by Lemma 2.3A) the last term in (5) maps $W(x,t) \in \overline{C}^{o}_{\alpha}$ continuously into $\overline{C}^{o}_{2+\alpha}$. Hence (5) also is a continuous linear map of $\overline{C}^{o}_{2+\alpha}$ into $\overline{C}^{o}_{2+\alpha}$.

Write (5) as

$$W(x,t) = q(x,t) + \int_{0}^{t}\int M(x,x';t,t')W(x',t')dx' \quad (9)$$

where

$$M(x,x';t,t') = \int G(x,y;t,t')b(y,t')k(x',y,t)dy.$$

The kernel $M(x,x';t,t')$ must also correspond to a continuous linear map of $\overline{C}^{o}_{2+\alpha}$ into $\overline{C}^{o}_{2+\alpha}$. By Theorem 3.1, and Lemma 2.3A

† See Lemma 2.3A for the definition of $\overline{C}^{o}_{2+\alpha}$.

$q(x,t)$ is in $\overline{C}^{o}_{2+\alpha}$ for some $\alpha > 0$ w.p.1. Then, the theory of Volterra integral equations asserts the existence of a $W(x,t) \in \overline{C}^{o}_{2+\alpha}$ satisfying (9) (and, hence, (5)) w.p.1. It also yields the representation (6)

The assertion concerning the Itô differential follows exactly as in the proof of Theorem 3.1. Q.E.D.

4. The Solution to the Optimum Control Problem

The solution is divided into four Theorems. Theorem 4.1 establishes some required properties of a partial differential integral equation (the analog of the Ricatti equation). Theorem 4.2 establishes a formula for the cost corresponding to a fixed control. Then, (Theorem 4.3) the usual dynamic programming technique of quasilinearization (or approximation in policy space) is applied to obtain a sequence of costs (and improved controls) which (Theorem 4.4) converges to the minimum cost (and optimal control).

<u>The adjoint of $\mathcal{L}$, operating on functions of x, is written as</u>

$$\mathcal{L}_x^* \xi(x) = \sum_{i,j} D_i D_j [a_{ij}(x,t)\xi(x)] - \sum D_i [d_i(x,t)\xi(x)]$$

$$+ c(x,t)\xi(x).$$

Define the Banach space $\hat{C}^o_{2+\alpha}$ of functions on $\hat{R} = \bar{D} \times \bar{D} \times [0,T]$ satisfying the condition that $f(x,y,t)$, $\mathcal{L}_x^* f(x,y,t)$ and $\mathcal{L}_y^* f(x,y,t) \to 0$ as $x \to \partial D$ or $y \to \partial D$, or $t \to T$, and with norm

$$\|f\|_{2+\alpha} = \|f\|_\alpha + \sum_i \|D_{x_i} f\|_\alpha + \sum_i \|D_{y_i} f\|_\alpha$$

$$+ \sum_{i,j} \|D_{x_i} D_{y_i} f\|_\alpha + \sum_{i,j} \|D_{x_i} D_{y_i} f\|_\alpha + \|f_t\|_\alpha$$

where

$$\|f\|_\alpha = \sup_{x,y,t\in\hat{R}} |f(x,y,t)| + \sup_{\substack{x,y,t\in\hat{R} \\ x',y',t'\in\hat{R}}} \frac{|f(x,y,t)-f(x',y',t')|}{|x-x'|^\alpha+|y-y'|^\alpha+|t-t'|^{\alpha/2}}$$

__Theorem__ 4.1. _Assume the conditions of Theorem_ 3.2, _and_ (E-7). _Let_ $Q(x,y,t)$ _be symmetric, Holder continuous on_ $\bar{D}\times\bar{D}\times[0,T]$ _and positive definite_† _for_ $t\in[0,T]$. _Let_ $Q(x,y,t)\to 0$ _as_ $x\to\partial D$ _or_ $y\to\partial D$. _Write_

$$\begin{aligned} & R_t(x,y,t) + (\mathcal{L}_x^*+\mathcal{L}_y^*)R(x,y,t) \\ & + \int b(v,t)[k(x,v,t)R(v,y,t) + k(y,v,t)R(x,v,t)]dv \qquad (10) \\ & = -Q(x,y,t). \end{aligned}$$

There is a unique symmetric (in x,y) _and continuous solution to_ (10) _which, in addition, is in_ $\hat{C}^o_{2+\alpha}$.

Proof. By (E-7), the adjoint operator and its Greens function are defined. The proof involves some standard calculations, similar to those of Sections 3-5, Chapter 1, [5], and most of the details are left to the reader. Consider first the adjoint equation (11), defined in $\bar{D}\times\bar{D}\times[0,T]$

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†By positive definiteness, we mean $\int Q(x,y,t)\varphi(x)\varphi(y)dxdy \geq m\int|\varphi(x)|^2dx$. The condition is not actually needed until Theorem 4.4.

$$\tilde{R}_t(x,y,t) + (\mathcal{L}_x^*+\mathcal{L}_y^*)\tilde{R}(x,y,t) = -Q(x,y,t) \tag{11}$$

with boundary conditions $\tilde{R}(x,y,t) \to 0$ as $x \to \partial D$ or $y \to \partial D$ or $t \to T$. The unique solution to (11) can be verified to be the symmetric function.[†,††]

$$\begin{aligned}\tilde{R}(x,y,t) &= \int_t^T ds\iint dx'dy' G(x',x;s,t)G(y',y;s,t)Q(x',y',s) \\ &= \int_t^T ds\iint dx'dy' G^*(x,x';t,s)G^*(y,y';t,s)Q(x',y',s).\end{aligned} \tag{12}$$

Write (12) as

$$R(x,y,t) = \int_t^T \int dsdx' G^*(x,x';t,s)h(y,x';t,s)$$

where

$$h(y,x';t,s) = \int G^*(y,y';t,s)Q(x',y',s)dy'.$$

$h(y,x';t,s)$ is uniformly Hölder continuous in y,x',t,s for $s \geqq t$, and $h(y,x';t,s) \to 0$ as $x \to \partial D$ or $y \to \partial D$. Let $t_o \leqq t$ and consider

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[†] Consider the differential equation (*) $\dot{P} = AP + PA' - Q$, with boundary condition $P(T) = 0$. Let $\Phi(t,s)$ be the fundamental matrix of $\dot{x} = Ax$. Then (*) has the solution (**) $P(t) = \int_t^T \Phi(t,s)Q(s)\Phi'(t,s)ds$. Note the similarity in form between (**) and (12). (Here ' denotes transpose.)

[††] Recall that $G_t^*(x,x';t,t') + \mathcal{L}_x^*G(x,x';t,t') = 0$ for $t < t'$.

$$\tilde{R}(x,y,t|t_o) = \int_t^T \int dsdx' G^*(x,x';t,s)h(y,x';t_o,s)$$

$\tilde{R}(x,y,t|t_o)$ is the solution to the adjoint equation $U_t + \mathcal{L}^* U = h$ (with $t = t_o$ in h) to which Lemma 2.3A is applicable.† Thus, for each fixed y, $\tilde{R}(x,y,t|t_o)$ and $\mathcal{L}_x^* \tilde{R}(x,y,t|t_o)$ tend to zero as $x \to \partial D$ or $t \to T$, by virtue of the properties of $Q(x,y,t)$; also $\tilde{R}(x,y,t|t_o)$ is symmetric in x,y. Since $\tilde{R}(x,y,t_o|t_o)$ and $\mathcal{L}_x^* \tilde{R}(x,y,t_o|t_o)$ tend to zero as $x \to \partial D$ or $y \to \partial D$, and t_o is arbitrary, we conclude that the terms $\tilde{R}(x,y,t|t) = \tilde{R}(x,y,t)$ and $\mathcal{L}_x^* \tilde{R}(x,y,t)$ and $\mathcal{L}_y^* \tilde{R}(x,y,t)$ are Holder continuous and tend to zero as $x \to \partial D$ or $y \to \partial D$ or $t \to T$. Now, to complete the proof that $\tilde{R}(x,y,t) \in \hat{C}^o_{2+\alpha}$ we need only show that $\tilde{R}_t(x,y,t)$ is Holder continuous. But this is true since $\tilde{R}(x,y,t|t_o)$ has Hölder continuous derivatives with respect to t and t_o in $t \geq t_o \geq 0$ (uniformly in x,y in $\bar{D} \times \bar{D}$).

For the rest of the proof, write (10) as the Volterra integral equation

$$R(x,y,t) = \tilde{R}(x,y,t) + \int_t^T ds \iint dx'dy' G^*(x,x';t,s) G^*(y,y';t,s) M(x',y',s) \tag{13}$$

where $M(x',y',s)$ is the integral term in the middle of (10) (with x',y',s substituted for x,y,t, resp.). If there is a solution of the desired form to ((14) is obtained by changing the order of integration in (13))

$$R(x,y,t) = \tilde{R}(x,y,t) + \int_t^T ds \iint R(v,w,s) K(x,y;v,w;t,s) dvdw + \int_t^T ds \iint R(v,w,s) K(y,x;w,v;t,s) dvdw, \tag{14}$$

† The boundary conditions are $U(x,t) = 0$ on $\bar{D} \times \{T\} + \partial D \times [0,T]$. If the time parameter is reversed (changing the terminal manifold $\bar{D} \times \{T\}$ to an initial manifold $\bar{D} \times \{0\}$) then, since $\mathcal{L}^*$ satisfies (E1) - (E3), Lemma 2.3A is applicable.

where

$$K(x,y;v,w;t,s) = b(v,s)G^*(y,w;t,s)\int G^*(x,x';t,s)k(x',v,s)dx',$$

then there is a solution of the desired form to (13). It can be verified that under the imposed conditions, the kernel K represents† a continuous linear map of $\hat{C}^o_{2+\alpha}$ into $\hat{C}_{2+\alpha}$.

Finally, it can be verified, via the theory of Volterra equations, that (14) does have a unique solution of the desired form. Q.E.D.

Let the control u be given by

$$u(x,s) = \int k^u(v,x,s)W(v,s)dv.$$

The the cost becomes

$$C^u(\varphi,t) = E^u_\varphi \int\int_t^T dxdydsW(x,s)W(y,s)Q^u(x,y,s)$$

where

$$Q^u(x,y,s) = S(x,y,s) + \int k^u(x,v,s)k^u(y,v,s)P(v,s)dv.$$

Recall that the system that we are concerned with is defined by Theorem 3.2 and has the Itô differential

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†The map is given by the sum of the two integrals of (14).

$$dW(x,t) = \mathcal{L}W(x,t)dt + \sigma(x,t)dz + b(x,t)u(x,t)dt,$$

and boundary condition $W(x,t) \to 0$, as $x \to \partial D$.

Theorem 4.2. *Assume the conditions of Theorem* 4.1. *Let* $W(x,t)$ *satisfy* (5), *with the control* $u(x,t)$ *given by* (4), *and let the initial condition* (*given at time* $t \in [0,T]$) $W(x,t) = \varphi(x)$ *satisfy the conditions on* $\varphi(x)$ *of Theorem* 3.1.

Suppose that $S(x,y,t), P(v,t), k^u(x,v,t)$ *and* $b(v,t)$ *are Holder continuous in their arguments and* $S(x,y,t)$ *and* $k^u(x,v,t)$ *tend to zero as* $x \to \partial D$ *or* $y \to \partial D$. *Let* $P(x,t) > 0$ *on* $\bar{D} \times [0,T]$ *and let* $S(x,y,t)$ *be symmetric and non-negative definite on* $\bar{D} \times \bar{D}$ *for each* t *in* $[0,T]$. *Then*

$$\begin{aligned} C^u(\varphi,t) &\equiv E^u_\varphi \int\int_t^T dxdydsQ^u(x,y,s)W(x,s)W(y,s) \\ &= p(t) + \int dxdyR^u(x,y,t)\varphi(x)\varphi(y). \end{aligned} \tag{15}$$

where

$$p(t) = \tfrac{1}{2}\int\int_t^T dxdydsR^u(x,y,s)\sigma(x,s)\sigma(y,s),$$

$R^u(x,y,s)$ *is the function introduced in Theorem* 4.1, *corresponding to* $Q(x,y,s) = Q^u(x,y,s)$.

Proof. The assumptions on $S(x,y,t), P(v,t)$, $k^u(x,v,t)$ and $b(v,t)$ guarantee that $Q^u(x,y,t)$ satisfies the conditions on $Q(x,y,t)$ in Theorem 4.1. For fixed x,y, the function

$$R^u(x,y,r)W(x,r)W(y,r) \equiv F(x,y,r)$$

has the Itô differential (Theorem 3.2) in $[0,T]$ (we use a version of $W(x,t)$ for which $\mathcal{L}W(x,t)$ is continuous w.p.1. - see Theorem 3.2)

$$\begin{aligned}
dF(x,y,r) &= R^u_r(x,y,r)W(x,r)W(y,r)dr + R^u(x,y,r)[dW(x,r)W(y,r) \\
&\quad + W(x,r)dW(y,r) + dW(x,r)dW(y,r)] \\
&= R^u_r(x,y,r)W(x,r)W(y,r)dr \\
&\quad + R^u(x,y,r)W(y,r)[\mathcal{L}W(x,r)dr + \sigma(x,r)dz + \\
&\qquad\qquad + b(x,r)(\textstyle\int W(v,r)k(v,x,r)dv)dr] \\
&\quad + R^u(x,y,r)W(x,r)[\mathcal{L}W(y,r)dr + \sigma(y,r)dz + \\
&\qquad\qquad + b(y,r)(\textstyle\int W(v,r)k(v,y,r)dv)dr] \\
&\quad + \tfrac{1}{2}R(x,y,r)\sigma(x,r)\sigma(y,r)dr.
\end{aligned}$$

This, together with $R^u(x,y,T) = 0$ implies (w.p.1 for each x,y,t) that

$$- F(x,y,t) = \int_t^T dF(x,y,r).$$

Furthermore,

$$-E_\varphi^u F(x,y,t) = -E_\varphi^u R^u(x,y,t)W(x,t)W(y,t) \tag{16}$$

$$= E_\varphi^u \int_t^T [R_r^u(x,y,r)W(x,r)W(y,r)$$

$$+ R^u(x,y,r)W(y,r)(\mathcal{L}W(x,r)dr + b(x,r)\int W(v,r)k(v,x,r)dv)$$

$$+ R^u(x,y,r)W(x,r)(\mathcal{L}W(y,r)dr + b(y,r)\int W(v,r)k(v,y,r)dv)]$$

$$+ \tfrac{1}{2}\int_t^T R^u(x,y,r)\sigma(x,r)\sigma(y,r)dr.$$

Each of the integrands on the right side of (16) can be defined be defined to be a measurable function of ω, x, y, r and absolutely integrable over $\Omega \times \bar{D} \times \bar{D} \times [t,T]$. Thus (16), together with Fubini's Theorem, implies that

$$-E_\varphi^u \int dxdy\, F(x,y,t) = -\int R^u(x,y,t)\varphi(x)\varphi(y)dxdy = \tag{17}$$

$$= E_\varphi^u \iint_t^T dxdydr[R_r^u(x,y,r)W(x,r)W(y,r)$$

$$+ R^u(x,y,r)W(y,r)(\mathcal{L}W(x,r) + b(x,r)\int W(v,r)k(v,x,r)dv)$$

$$+ R^u(x,y,r)W(x,r)(\mathcal{L}W(y,r) + b(y,r)\int W(v,r)k(v,y,r)dv)]$$

$$+ p(t).$$

Now $W(x,t)$ (w.p.1.) and $R^u(x,y,t)$ (for each fixed $y \in \bar{D}$) are continuous and have uniformly continuous first and second derivatives, with respect to the x_i, in the domain $\bar{R}$. Also $W(x,t)$ (w.p.1.) and $R^u(x,y,t)$ tend to zero as $x \to \partial D$. Thus, upon partially integrating and using Greens identity to eliminate the boundary in-

integrals which are obtained (which are zero, owing to the first two sentences of this paragraph), we get (for ω not in some null set)

$$\iint_t^T \mathrm{dxdydr}\ R^u(x,y,r)W(y,r)\mathcal{L}W(x,r)$$

$$= \iint_t^T \mathrm{dxdydr}\ \mathcal{L}_x^* R^u(x,y,r)W(y,r)W(x,r).$$

Substituting this in (17), and using the symmetry of $R(x,y,r)$, yields, after another change in the order of integration

$$-\int R^u(x,y,t)\varphi(x)\varphi(y)\mathrm{dxdy} =$$

$$= E_\varphi^u \iint_t^T \mathrm{dxdydr}\ W(x,r)W(y,r)[R_r^u(x,y,r) + (\mathcal{L}_x^* + \mathcal{L}_y^*)R^u(x,y,r)$$

$$+ \int b(x',r)k(x,x',r)R^u(x',y,r)dx' + \tag{18}$$

$$\int b(x',r)k(y,x',r)R^u(x,x',r)dx'] + p(t).$$

Finally, using the relation (10) in (18) yields (15). Q.E.D.

Write $C^u(W(x,t),t)$ for the function $C^u(\varphi,t)$ with $W(x,t)$ substituted for $\varphi(t)$. Write $d^v C^u(W(x,t),t)$ for the Itô differential of the cost (corresponding to control u) but where, in the expression for $dW(x,t)$, a control $v(x,t)$ replaces the control $u(x,t)$.

Theorem 4.3. Let $u(x,t)$ be a given control (then $k^u(x,y,t)$ is given) and assume the other conditions of Theorem 4.2. Let $\tilde{u}(x,t)$ be the function $v(x,t)$ which minimizes

$$E_\varphi^v \int_t^T d^v C^u(W(x,s),s) + E_\varphi^v \int_t^T \int dxdydsS(x,y,s)W(x,s)W(y,s) + E_\varphi^v \int_t^T \int dxdsP(x,s)v^2(x,s) \tag{19}$$

<u>or, equivalently, which minimizes</u>

$$E_\varphi^v \int dxdy\ R^u(x,y,s)[W(y,s)b(x,s)v(x,s) + W(x,s)b(y,s)v(y,s)] + E_\varphi^v \int dxP(x,s)v^2(x,s). \tag{20}$$

<u>Then</u>

$$C^{\tilde{u}}(\varphi,t) \leq C^u(\varphi,t) \tag{21}$$

$\tilde{u}(x,s)$ <u>is given by</u>

$$\tilde{u}(x,s) = -b(x,s)\int \frac{R^u(x,y,s)}{P(x,s)} W(y,s)dy \tag{22}$$

<u>and</u>

$$k^{\tilde{u}}(g,x,s) = -b(x,s)R^u(x,g,s)/P(x,s) \tag{23}$$

<u>and the corresponding</u> $R^{\tilde{u}}(x,y,t)$ <u>satisfies the conditions on the</u> $R^u(x,y,t)$ <u>of Theorem</u> 4.2. <u>Also</u>

$$\int \tilde{R}^u(x,y,t)\varphi(x)\varphi(y)dxdy \leq \int R^u(x,y,t)\varphi(x)\varphi(y)dxdy \tag{24}$$

for any bounded and measurable function $\varphi(x)$.

Proof. By Theorem 4.2, (19) is non-negative and equals zero when $v(x,t) = u(x,t)$. Then any minimizing $v(x,t)$, (provided that the corresponding integrals of (19) exist) must leave (19) non-positive. This, together with the facts that $C^v(\varphi,t)$ is the sum of the last two integrals in (19) and that the first integral of (19) equals $-C^u(\varphi,t)$, implies (21). The $v(x,t)$ minimizing (19) is the $v(x,t)$ which minimizes

$$E_\varphi^v \int_t^T \int dxdyds[R_s^u(x,y,s)W(x,s)W(y,s) + R^u(x,y,s)W(y,s)(\mathcal{L}W(x,s)+b(x,s)v(x,s))$$

$$+ R^u(x,y,s)W(x,s)(\mathcal{L}W(y,s)+b(y,s)v(y,s))]$$

$$+ E_\varphi^v \int_t^T \int dxdydsS(x,y,s)W(x,s)W(y,s) + E_\varphi^v \int_t^T \int dxdsP(x,s)v^2(x,s)$$

or, equivalently, which minimizes (20). The $v(x,t)$ minimizing (20) is given by (22). The statement below (23) is easily established (via Theorem 4.2) since the $\tilde{k}^u(y,x,t)$ of (23) satisfies the conditions on $k^v(y,x,t)$ in the Hypothesis of Theorem 4.2. (24) is valid for all doubly differential functions $\varphi(x)$ which are zero on ∂D since

$$C^v(\varphi,t) = \int dxdyR^v(x,y,t)\varphi(x)\varphi(y) + \tfrac{1}{2}\iint_t^T dxdydrR^v(x,y,s)\sigma(x,s)\sigma(y,s)$$

$$\leq \int dxdyR^u(x,y,t)\varphi(x)\varphi(y) + \tfrac{1}{2}\iint_t^T dxdydsR^u(x,y,s)\sigma(x,s)\sigma(y,s)$$

for all such $\varphi(x)$. Hence (24) is valid for all functions which are (almost everywhere) pointwise limits of a bounded sequence of such $\varphi(x)$. Q.E.D.

Theorem 4.4 is the optimality theorem. Let k^n, Q^n and R^n correspond to k^{u_n}, Q^{u_n} and R^{u_n}, resp.

Theorem 4.4. _Let_ $u_o(x,t)$ _be given and let the corresponding_ $k^o(y,x,t)$ _satisfy the conditions on_ $k^u(y,x,t)$ _in Theorem_ 4.2. _Suppose the other conditions of Theorem_ 4.2 _hold. Define_ $u_n(x,t)$ _from_ $u_{n-1}(x,t)$, $n = 1,\ldots,$ _via the procedure of Theorem_ 4.3. _Then_ $R^n(x,y,t)$ _converges pointwise (almost everywhere) to an_ $R(x,y,t)$ _which satisfies the conditions of Theorem_ 4.1. _The control_ $u(x,t)$ _corresponding to_ $R(x,y,t)$ _via_ (25) (see (22))

$$\begin{aligned} u(x,s) &= -b(x,s)\int \frac{R(x,y,s)W(y,s)}{P(x,s)}dy \\ &\equiv \int k(y,x,s)W(y,s)dy \end{aligned} \tag{25}$$

is optimal in that $C^u(\varphi,t) \leq C^{\bar{v}}(\varphi,t)$ _for any other control_

$$\bar{v}(x,s) = \int k^{\bar{v}}(y,x,s)W(y,s)dy \tag{25a}$$

where $k^v(y,x,t)$ satisfies the condition on the $k^u(y,x,t)$ in Theorem 4.2. $R(x,y,t)$ also satisfies the boundary conditions on the $R^u(x,y,t)$ of Theorem 4.2, and the 'Ricatti' equation

$$\begin{aligned} &R_t(x,y,t) + (\mathcal{L}_x^* + \mathcal{L}_y^*)R(x,y,t) \\ &+ \int b(v,t)[k(x,v,t)R(v,y,t) + k(y,v,t)R(x,v,t)]dv \\ &= -Q(x,y,t) \end{aligned} \tag{26}$$

where $k(y,x,t)$ is given by (25) $k(x,v,t) = -b(v,t)R(v,x,t)/P(v,t)$ and also

$$Q(x,y,t) = S(x,y,t) + \int k(x,v,t)k(y,v,t)P(v,s)dv. \tag{27}$$

Proof. The proof is divided into three steps. First, we show that $R^n(x,y,t)$ converges (almost everywhere) to some function $R(x,y,t)$; second that $R(x,y,t)$ is smooth and satisfies (26), and third, that $R(x,y,t)$ corresponds to the optimal control. By (24) (where u and $\tilde{u}$ are replaced by u_n and u_{n+1}) and the non-negative definiteness of the $R^i(x,y,t)$,

$$\int dxdy[R^n(x,y,t)-R^{n+1}(x,y,t)]\varphi(x)\varphi(y) \geq 0 \tag{28}$$

for any bounded measurable $\varphi(x)$. Also the $R^n(x,y,t)$ are continuous in $\bar{D} \times \bar{D} \times [0,T]$. Hence, the diagonal values $R^n(x,x,t)$ are non-negative and non-increasing as n increases, and $R^n(x,x,t) \downarrow R(x,x,t)$ (almost everywhere) for some function $R(x,x,t)$. This together with $\max_{x,y} |R^n(x,y,t)| \leq \max_x |R^n(x,x,t)|$, implies that the $R^n(x,y,t)$ are uniformly bounded. In fact, the pointwise convergence implies that the diagonal values converge 'almost uniformly' in the sense that for any fixed $\epsilon > 0$ there is a function $\alpha(N,\epsilon)$ tending to zero as $N \to \infty$ and a set $S_\alpha \subset \bar{D}$ with Lebesgue measure $\alpha(N,\epsilon)$ so that, for any $m,n > N$

$$0 \leq R^n(x,x,t) - R^m(x,x,t) < \epsilon \qquad (29)$$

on $\bar{D} - S_\alpha$. Next, suppose that, for some $m > n > N$

$$R^m(x',x'',t) - R^n(x',x'',t) > 2\epsilon, \qquad (29a)$$

on some $(x',x'') \in (\bar{D}-S_\alpha) \times (\bar{D}-S_\alpha)$. Then, by continuity and symmetry, there are neighborhoods A',A'' of x',x'', resp (A',A'' are assumed to be in $\bar{D}-S_\alpha$) so that (29a) and (29) hold on $A' \times A'' \cup A'' \times A'$. <u>Let</u> $I(A)(x)$ <u>be the characteristic function of the set</u> A. Set $\varphi(x) = I(A' \cup A'')(x)$. Then using this and the diagonal (29) bound in (28) gives

$$\int [R^n(x,y,t)-R^m(x,y,t)] I(A' \cup A'')(x) \cdot I(A' \cup A'')(y)\, dx\, dy$$

$$\leq (2\epsilon - 4\epsilon)\mu(A')\mu(A'') < 0$$

$(u(\cdot))$ is Lebesgue measure on $\bar{D}$, a contradiction. Since α can be made arbitrarily small by increasing N, we conclude that $R^n(x,y,t)$ converges almost everywhere to a function $R(x,y,t)$. Furthermore, it is clear that $R(x,y,t)$ is symmetric, measurable and bounded almost everywhere by $r(x,y,t)$, where $r(x,y,t)$ is some function which tends to zero as $x \to \partial D$ or $y \to \partial D$.

To continue, we use the representation (see Theorems 4.1 and 4.3 for terminology)

$$\begin{aligned} R^{n+1}(x,y,t) = {} & \tilde{R}^{n+1}(x,y,t) + \\ & + \int_t^T \int ds\,dv\,dw\,K^{n+1}(x,y;v,w;t,s)R^n(v,w,s) \\ & + \int_t^T \int ds\,dv\,dw\,K^{n+1}(y,x;w,v;t,s)R^n(v,w,s) \end{aligned} \tag{30}$$

$$\tilde{R}^{n+1}(x,y,t) = \int_t^T \int ds\,dx'\,dy'\,G^*(x,x';t,s)G^*(y,y';t,s)Q^{n+1}(x',y',s) \tag{31}$$

$$Q^{n+1}(x',y',s) = S(x',y',s) + \int k^{n+1}(x',v,s)k^{n+1}(y',v,s)P(v,s)dv \tag{32}$$

$$k^{n+1}(x',v,s) = -b(v,s)R^n(v,x',s)/P(v,s) \tag{33}$$

$$K^{n+1}(x,y;v,w;t,s) = \frac{-b^2(v,s)}{P(v,s)}G^*(y,w;t,s)\int G^*(x,x';t,s)R^n(v,x',s)dx'. \tag{34}$$

The left side of (30) tends (almost everywhere) to $R(x,y,t)$. Similarly (and we omit the uninteresting details) the limit of each sequence of integrals can be replaced by the integral of the (almost everywhere) limit of the integrands.

Thus, a.e.

$$R(x,y,t) = \tilde{R}(x,y,t) + \int_t^T \int dsdvdwK(x,y;v,w;t,s)R(v,w,s)$$
$$+ \int_t^T \int dsdvdwK(y,x;w,v;t,s)R(v,w,s), \tag{35}$$

where the kernel K is given by (34) with $R(v,x',s)$ replacing $R^n(v,x',s)$.

Consider the $\tilde{R}(x,y,t)$ term in (35). Since $Q(x,y,t)$ is symmetric, bounded, measurable and tends to zero (almost everywhere) as $x \to \partial D$ or $y \to \partial D$, $\tilde{R}(x,y,t)$ is Holder continuous and tends to zero as $x \to \partial D$ or $y \to \partial D$ or $t \to T$. If a member of this latter class is substituted for the $Q(x,y,t)$, then $\tilde{R}(x,y,t) \in \hat{C}^o_{2+\alpha}$. Similarly the map K (the sum of the integrals in (35)) takes $R(x,y,s)$ into a Holder continuous function which tends to zero as $x \to \partial D$, $y \to \partial D$ or $t \to T$. If a member of this class is substituted for the $R(x,y,s)$ in the kernel K, then the sum of the integrals in (35) is in $\hat{C}^o_{2+\alpha}$. (Recall the identical assertion in the proof of Theorem 4.1.) These considerations imply that $R(x,y,t)$ is indeed in $\hat{C}^o_{2+\alpha}$.

Upon differentiating (35) and using $G_t(x,x';t,s) = -\mathcal{L}^*_x G(x,x';t,s)$ we get (26).

Now, note that the $u(x,t)$ in (25) is the $v(x,t)$ which minimizes (19) and (20). Thus, letting $\bar{v}(x,t)$ be a control of the form (25a), (19) yields

$$E_\varphi^u \int_t^T d^u C^u(W(x,s),s) + E_\varphi^u \int_t^T \int dsdxdyS(x,y,s)W(x,s)W(y,s)$$

$$+ E_\varphi^u \int_t^T \int dsdxP(x,s)u^2(x,s)$$

$$\leq E_\varphi^{\bar{v}} \int_t^T d^{\bar{v}} C^u(W(x,s),s) + E_\varphi^{\bar{v}} \int_t^T \int dsdxdyS(x,y,s)W(x,s)W(y,s) \tag{36}$$

$$+ E_\varphi^{\bar{v}} \int_t^T \int dsdxP(x,s)\bar{v}^2(x,s).$$

Since the first terms on the left and right of (36) are equal, (36) implies $C^u(\varphi,t) \leq C^{\bar{v}}(\varphi,t)$. Q.E.D.

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